Kohn's theorem in a superfluid Fermi gas with a Feshbach resonance

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Abstract

We investigate the dipole mode in a superfluid gas of Fermi atoms trapped in a harmonic potential. According to Kohn's theorem, the frequency of this collective mode is not affected by an interaction between the atoms and is always equal to the trap frequency. This remarkable property, however, does not necessarily hold in an approximate theory. We explicitly prove that the Hartree-Fock-Bogoliubov generalized random phase approximation (HFB-GRPA), including a coupling between fluctuations in the density and Cooper channels, is consistent with both Kohn's theorem as well as Goldstone's theorem. This proof can be immediately extended to the strong-coupling superfluid theory developed by Noziéres and Schmitt-Rink (NSR), where the effect of superfluid fluctuations is included within the Gaussian level. As a result, the NSR-GRPA formalism can be used to study collective modes in the BCS-BEC crossover region in a manner which is consistent with Kohn's theorem. We also include the effect of a Feshbach resonance and a condensate of the associated molecular bound states. A detailed discussion is given of the unusual nature of the Kohn mode eigenfunctions in a Fermi superfluid, in the presence and absence of a Feshbach resonance. When the molecular bosons feel a different trap frequency from the Fermi atoms, the dipole frequency is shown to *depend* on the strength of effective interaction associated with the Feshbach resonance.

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I. INTRODUCTION

Among various collective excitations in an atomic gas trapped in a harmonic potential, the dipole mode has the unique property that its frequency is always equal to the trap frequency, irrespective of the interaction between the atoms. This remarkable property was originally proved by Kohn in the context of the cyclotron frequency of electrons in metals[1], and later it was extended to an excitation spectrum of electrons in a quantum well produced in $Al_xGa_{1-x}As$[2, 3]. Recently, Kohn's theorem has been extensively discussed in the context of Bose-Einstein condensation (BEC) of trapped atomic gases[4].

Kohn's theorem is a direct consequence of the translational invariance of particle-particle interaction term, $U_{\rm int} \equiv \sum_{i<j} u(\mathbf{r}_i - \mathbf{r}_j)$. To show this, we consider an N-body system described by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + U_{\rm int} + \frac{1}{2} m \sum_{i=1}^{N} [\Omega_x^2 x_i^2 + \Omega_y^2 y_i^2 + \Omega_z^2 z_i^2]. \tag{1.1}$$

It is easy to see that the operators $\hat{P}_\alpha \equiv \sum_{i=1}^{N} [m\Omega_\alpha \hat{r}_{\alpha,i} - i\hat{p}_{\alpha,i}]$ $(\alpha = x, y, z)$ satisfy the commutation relations $[H, \hat{P}_\alpha] = \Omega_\alpha \hat{P}_\alpha$[2]. We note that the fact that $[\hat{P}_\alpha, U_{\rm int}] = 0$ is crucial to obtain this relation. If $|\Psi_0\rangle$ is the ground state wave-function with energy E_0, the three excited states generated by $\hat{P}_\alpha$, $|\Psi_1\rangle_\alpha \equiv \hat{P}_\alpha |\Psi_0\rangle$ $(\alpha = x, y, z)$, are eigenstates with energies $E_\alpha = \Omega_\alpha + E_0$, with $U_{\rm int}$ having no effect. These excited states are referred to collectively as the "Kohn modes" in the theoretical literature (the "sloshing modes" in the recent cold atom experimental literature).

While Kohn's theorem is exact, it is not a trivial problem as to whether or not it holds when the interaction is treated approximately. For example, in the mean-field approximation ($U_{\rm int} \to U_{\rm MF} = \sum_i \int d\mathbf{r}' u(\mathbf{r}_i - \mathbf{r}') n(\mathbf{r}')$, where $n(\mathbf{r}')$ is the particle density), the translational invariance of the original (exact) $U_{\rm int}$ is broken, which leads to $[\hat{P}_\alpha, U_{\rm MF}] \neq 0$. Since, in most cases, we cannot treat many-body effects exactly, it is an important goal to obtain an approximation consistent with Kohn's theorem. The nature of the Kohn mode has been extensively studied in trapped Bose gases[4, 5, 6, 7, 8, 9]. In particular, Fetter et. al.[8] and Reidl et. al.[9] proved that, in trapped Bose gases, the Hartree-Fock random phase approximation (HF-RPA) is consistent with this theorem at $T = 0$ and at $T > 0$, respectively.

This problem is particularly crucial for fermion superfluidity of atoms in a trap, which

is a topic of great current interest, because the BCS pairing *approximation* ($U_{\rm BCS} \equiv -U\Psi^\dagger_\uparrow(\mathbf{r})\Psi^\dagger_\downarrow(\mathbf{r})\Psi_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}) \rightarrow \Delta(\mathbf{r})[\Psi^\dagger_\uparrow(\mathbf{r})\Psi^\dagger_\downarrow(\mathbf{r}) + h.c.]$, where $\Psi_\sigma(\mathbf{r})$ is a fermion field operator and $\Delta(\mathbf{r})$ is the Cooper-pair order parameter) is almost always used in the study of the BCS superfluid phase. In a trap, the $\mathbf{r}$-dependent order parameter $\Delta(\mathbf{r})$ destroys the translational invariance of the pairing interaction. Besides this, in a trapped gas of Fermi atoms, we need to take into account a Feshbach resonance and the associated molecules[10, 11]. The enhanced pairing interaction mediated by the molecules can be used to study superfluidity[12, 13, 14, 15, 16, 17, 18] in atomic Fermi gases, such as ^{40}K and ^{6}Li[19, 20, 21, 22]. In addition, since the strength of this effective interaction is tunable by an external magnetic field, one can probe the BCS-BEC crossover phenomenon[23, 24, 25, 26], where the superfluidity continuously changes from a BCS-type to a BEC of composite bosons[15, 16, 17, 18]. Very recently, superfluidity both in the BEC regime[27] and in the crossover regime[28, 29, 30, 31] has been observed in ^{40}K and ^{6}Li. In dealing with such systems, one must ensure that approximate theories are consistent with Kohn's theorem.

In this paper, we give an explicit formal proof that the Hartree-Fock-Bogoliubov generalized random phase approximation (HFB-GRPA), including a coupling between density fluctuations with superfluid fluctuations, is consistent with Kohn's theorem at all temperatures. We prove that this is also valid for the BCS-BEC crossover. It is also valid for the case when one includes fluctuations around the mean-field approximation, such as first done by Nozières and Schmitt-Rink (NSR)[24] at $T_{\rm c}$. This proof can be easily extended to deal with systems with a Feshbach resonance. Thus our results show that the NSR-GRPA formalism can be safely used to study response functions and collective modes in the whole region of BCS-BEC crossover, without fear of the breakdown of Kohn's theorem. We use the same formalism to explicitly verify that a zero frequency Goldstone sound mode arises associated with the Bose broken symmetry.

We also show that, as expected, Kohn's theorem is no longer valid in the presence of Feshbach resonance if the molecular bosons feel a different trap frequency $\Omega_{\rm B}$ from the trap frequency Ω_F which the Fermi atoms feel. In this case, the dipole mode frequency does *depend* on the strength of the effective interaction as we go through the BCS-BEC crossover regime.

We give the explicit forms for the Kohn mode eigenstates in the BCS-BEC crossover region, without and with a Feshbach resonance. In the case of a Feshbach resonance in the

crossover region, we show that the Kohn mode eigenstate is not simply a rigid center of mass oscillation of the static condensate profile.

This paper is organized as follows. In Sec. II, we prove that HFB-GRPA is consistent with Kohn's theorem at all temperatures, in the absence of the Feshbach resonance. The extension of this proof to the NSR-GRPA treatment of fluctuations is also discussed. In Sec. III, we generalize the proof to include the appearance of molecules associated with a Feshbach resonance. We also consider the case when the Fermi atoms and quasi-molecules feel different trap frequencies. Throughout this paper, we set $\hbar = 1$ for simplicity.

II. KOHN'S THEOREM IN THE BCS APPROXIMATION

A. Hartree-Fock Bogoliubov approximation

We consider a two-component Fermi gas trapped in a harmonic potential described by the Hamiltonian,

$$H = \sum_\sigma \int d\mathbf{r}\Psi^\dagger_\sigma(\mathbf{r})[-\frac{\nabla^2}{2m} + V_{\rm trap} - \mu]\Psi_\sigma(\mathbf{r}) - U\int d\mathbf{r}\Psi^\dagger_\uparrow(\mathbf{r})\Psi^\dagger_\downarrow(\mathbf{r})\Psi_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}), \quad (2.1)$$

where $\Psi_\sigma(\mathbf{r})$ is a fermion field operator with pseudo-spin $\sigma = \uparrow, \downarrow$, and μ is the chemical potential. U is an s-wave pairing interaction, and $V_{\rm trap}(\mathbf{r})$ is an anisotropic harmonic trap potential given by

$$V_{\rm trap}(\mathbf{r}) = \sum_{\alpha=x,y,z}\frac{1}{2}m\Omega_\alpha^2 r_\alpha^2. \quad (2.2)$$

The proof of Kohn's theorem explained in the introduction can be easily extended to the second quantized Hamiltonian in Eq. (2.1). When we define

$$\hat{P}_\alpha \equiv \sum_\sigma \int d\mathbf{r}\Psi^\dagger_\sigma(\mathbf{r})[m\Omega_\alpha\hat{r}_\alpha - i\hat{p}_\alpha]\Psi_\sigma(\mathbf{r}) \quad (\alpha = x, y, z), \quad (2.3)$$

this operator satisfies $[H, \hat{P}_\alpha] = \Omega_\alpha\hat{P}_\alpha$. Thus, the state $\hat{P}_\alpha|\Psi_0\rangle$ has the excitation energy Ω_α, if $|\Psi_0\rangle$ is the ground state wave-function.

In the Hartree-Fock-Bogoliubov (HFB) approximation, Eq. (2.1) reduces to

$$H_{\rm HFB} = \sum_\sigma \int d\mathbf{r}\Psi^\dagger_\sigma(\mathbf{r})\Big[\hat{h}_0(\mathbf{r}) - \frac{U}{2}n(\mathbf{r})\Big]\Psi_\sigma(\mathbf{r}) + \int d\mathbf{r}[\Delta(\mathbf{r})\Psi^\dagger_\uparrow(\mathbf{r})\Psi^\dagger_\downarrow(\mathbf{r}) + {\rm h.c.}], \quad (2.4)$$

where $\hat{h}_0(\mathbf{r}) = -\frac{\nabla^2}{2m} + V_{\rm trap}(\mathbf{r}) - \mu$ is the Hamiltonian for non-interacting Fermi atoms. $H_{\rm HFB}$ includes two mean fields, involving the local density of Fermi atoms $n(\mathbf{r}) \equiv \sum_\sigma\langle\Psi^\dagger_\sigma(\mathbf{r})\Psi_\sigma(\mathbf{r})\rangle$

as well as the off-diagonal field associated with the Cooper-pair order parameter $\Delta(\mathbf{r}) \equiv -U\langle\Psi_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r})\rangle$[32]. In the well-known Nambu representation, $H_{\rm HFB}$ is conveniently written as[33]

$$H_{\rm HFB} = \int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\Big[\hat{h}_0(\mathbf{r})\sigma_3 - \frac{U}{2}n(\mathbf{r})\sigma_3 + \Delta(\mathbf{r})\sigma_1\Big]\hat{\Psi}(\mathbf{r}), \tag{2.5}$$

where $\hat{\Psi}^\dagger(\mathbf{r}) \equiv (\Psi_\uparrow^\dagger(\mathbf{r}), \Psi_\downarrow(\mathbf{r}))$ and

$$\hat{\Psi}(\mathbf{r}) \equiv \begin{pmatrix} \Psi_\uparrow(\mathbf{r}) \\ \Psi_\downarrow^\dagger(\mathbf{r}) \end{pmatrix} \tag{2.6}$$

are the two-component Nambu field operators, and σ_α ($\alpha = 1, 2, 3$) are the Pauli matrices. In Eq. (2.5), we take the order parameter $\Delta(\mathbf{r})$ real and proportional to the σ_1-component. This choice is always possible, in the absence of supercurrents or vortices.

The HFB Hamiltonian in Eq. (2.5) can be diagonalized using the solution of the well-known Bogoliubov de-Gennes (BdG) equations,

$$\hat{h}\Psi_n(\mathbf{r}) = E_n\Psi_n(\mathbf{r}), \tag{2.7}$$

where

$$\hat{h} \equiv \hat{h}_0(\mathbf{r})\sigma_3 - \frac{U}{2}n(\mathbf{r})\sigma_3 + \Delta(\mathbf{r})\sigma_1. \tag{2.8}$$

The energy E_n of the two-component wave-function

$$\Psi_n(\mathbf{r}) = \begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix} \tag{2.9}$$

can be both positive and negative. Actually, a negative energy state $\Psi_{n<0}$ ($-E_n < 0$) is related to a positive energy state $\Psi_{n>0}$ ($E_n > 0$) by

$$\Psi_{n<0} = i\sigma_2\Psi_{n>0}. \tag{2.10}$$

Thus, we need only solve for the positive energy solutions of the BdG equations.

The Bogoliubov transformation[34] is given by

$$\hat{\Psi}(\mathbf{r}) = \sum_{E_n>0} [\Psi_n(\mathbf{r})\gamma_{n\uparrow} + i\sigma_2\Psi_n(\mathbf{r})\gamma_{n\downarrow}^\dagger], \tag{2.11}$$

in which case Eq. (2.4) can be diagonalized as

$$H_{\rm HFB} = \sum_{E_n>0,\sigma} E_n\gamma_{n\sigma}^\dagger\gamma_{n\sigma}. \tag{2.12}$$

Here $\gamma_{n\sigma}$ is the annihilation operator of a Bogoliubov quasi-particle. When we define the fermion operators $\gamma_{n>0} \equiv \gamma_{n\uparrow}$ ($E_n > 0$) and $\gamma_{n<0} \equiv \gamma^\dagger_{n\downarrow}$ ($E_n < 0$), Eqs. (2.11) and (2.12) are rewritten, respectively, as

$$\hat{\Psi}(\mathbf{r}) = \sum_n \Psi_n(\mathbf{r})\hat{\gamma}_n, \tag{2.13}$$

$$H_{\mathrm{HFB}} = \sum_n E_n \hat{\gamma}^\dagger_n \hat{\gamma}_n, \tag{2.14}$$

where the summations in Eqs. (2.13) and (2.14) are taken over all the eigenstates with both positive and negative energies. In the following sections, we use the Bogoliubov transformation given by Eq. (2.13).

The two mean-fields $\Delta(\mathbf{r})$ and $n(\mathbf{r})$ are determined self-consistently by

$$\Delta(\mathbf{r}) = \frac{U}{2} \sum_{E_n>0}^{\omega_c} \Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})[1-2f_n], \tag{2.15}$$

$$n(\mathbf{r}) = -\sum_{E_n>0} \Psi^\dagger_n(\mathbf{r})\sigma_3\Psi_n(\mathbf{r})[1-2f_n] + \delta(0). \tag{2.16}$$

Here f_n is the Fermi distribution function with energy E_n. As usual, we need a cutoff ω_c in the energy-summation of the gap equation (2.15). The divergent term $\delta(0)$ in Eq. (2.16) is ultimately canceled out by the divergent term involved in the first term.

B. Generalized random phase approximation (GRPA)

The static HFB Hamiltonian in Eq. (2.4) neglects fluctuations in both the density and Cooper channels. These fluctuation effects can be included by including interactions left out in the HFB approximation. To describe such interactions, it is convenient to introduce a *generalized* density operator $\hat{\rho}_\alpha \equiv \hat{\Psi}^\dagger(\mathbf{r})\sigma_\alpha\hat{\Psi}(\mathbf{r})$, where $\alpha = 1, 2, 3$ represent the amplitude fluctuations of the order parameter, the phase fluctuations of the order parameter, and density fluctuations, respectively[35]. Then interactions involving fluctuations in Eq. (2.1) can be written as[17, 36, 37]

$$U^{\mathrm{FL}}_\alpha \equiv -\frac{U}{4}\int d\mathbf{r}\hat{\rho}_\alpha(\mathbf{r})\hat{\rho}_\alpha(\mathbf{r}) \quad (\alpha = 1, 2, 3). \tag{2.17}$$

Here U^{FL}_1 and U^{FL}_2 are interactions in the Cooper channel, while the interaction in the density channel is given by U^{FL}_3.

In the superfluid phase, density fluctuations couple with superfluid fluctuations through the Josephson effect. The *generalized* random phase approximation (GRPA) is a RPA kind

of approximation which includes this additional coupling[36]. When we use the GRPA to treat $U_j^{\rm FL}$ in Eq. (2.17) in a linear response calculation, we find that the response in the generalized density is described by

$$\delta\rho_\alpha(\mathbf{r},\omega) = -\frac{U}{2}\sum_{\beta=1}^{3}\int d\mathbf{r}'\Pi^0_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)\delta\rho_\beta(\mathbf{r}',\omega) \quad (\alpha=1,2,3). \tag{2.18}$$

Here $\Pi^0_{\alpha\beta}$ is the zero-th order *generalized* density correlation function defined by[36]

$$\Pi^0_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega) = -i\int_0^\infty dt e^{i\omega t}\langle[\hat{\rho}_\alpha(\mathbf{r},t),\hat{\rho}_\beta(\mathbf{r}',0)]\rangle. \tag{2.19}$$

As we discuss elsewhere[17], the ordinary density correlation function is Π^0_{33}, while Π^0_{11} and Π^0_{22} describe amplitude and phase fluctuations of the order parameter, respectively. The off-diagonal components of Eq. (2.19) represent coupling between these fluctuations. For example, Π^0_{23} expresses a phase-density coupling, originating from the Josephson effect. Thus, Eq. (2.18) describes a collective mode in terms of the amplitude ($\delta\rho_1$) and phase ($\delta\rho_2$) oscillations in the Cooper-channel, as well as the ordinary density oscillation ($\delta\rho_3$).

In a uniform gas, where we can use $\delta\rho_\alpha(\mathbf{r},\omega) = e^{i\mathbf{q}\cdot\mathbf{r}}\delta\rho_\alpha(\mathbf{q},\omega)$, Eq. (2.18) reduces to the 3×3 matrix equation

$$\Big[1-\frac{U}{2}\hat{\Pi}^0(\mathbf{q},\omega)\Big]\begin{pmatrix}\delta\rho_1(\mathbf{q},\omega)\\ \delta\rho_2(\mathbf{q},\omega)\\ \delta\rho_3(\mathbf{q},\omega)\end{pmatrix} = 0. \tag{2.20}$$

Here $\hat{\Pi}^0(\mathbf{q},\omega) \equiv \{\Pi^0_{\alpha\beta}(\mathbf{q},\omega)\}$ $(\alpha,\beta=1,2,3)$ is the 3×3-matrix correlation function in momentum space (The detailed expressions for $\Pi^0_{\alpha\beta}(\mathbf{q},\omega)$ are given in Ref. [17].). The solution of the 3×3 matrix equation in Eq. (2.20) can be shown to be the same as the poles of the (3×3 matrix) GRPA density correlation function, $\hat{\Pi}(\mathbf{q},\omega) \equiv [1-\frac{U}{2}\hat{\Pi}^0(\mathbf{q},\omega)]^{-1}\hat{\Pi}^0(\mathbf{q},\omega)$[17, 38].

In a harmonic trap, $\Pi^0_{\alpha\beta}(\mathbf{r},\mathbf{r}',\omega)$ can be calculated from the analytic continuation of the corresponding two-particle *thermal* Green's functions,

$$\tilde{\Pi}^0_{\alpha\beta}(\mathbf{r},\mathbf{r}',i\nu_n) = \frac{1}{\beta}\sum_{\omega_l}{\rm Tr}\Big[\sigma_\alpha\hat{G}(\mathbf{r},\mathbf{r}',i\omega_l+i\nu_n)\sigma_\beta\hat{G}(\mathbf{r}',\mathbf{r},i\omega_l)\Big], \tag{2.21}$$

where $i\omega_l$ and $i\nu_n$ are the usual fermion and boson Matsubara frequencies, respectively. $\hat{G}(\mathbf{r},\mathbf{r}',i\omega_l)$ is the 2×2-matrix single-particle thermal Green's function, given by

$$\hat{G}(\mathbf{r},\mathbf{r}',i\omega_l) = \sum_n \frac{\Psi_n(\mathbf{r})\Psi_n^\dagger(\mathbf{r}')}{i\omega_l - E_n}, \tag{2.22}$$

where $\Psi_n(\mathbf{r})$ are the Nambu eigenstate spinors defined in Eq. (2.7). After doing the $i\omega_l$-summation in Eq. (2.21), we make the usual analytic continuation $i\nu_n \to \omega_+ \equiv \omega + i\delta$. The resulting diagonal correlation function is given by

$$\Pi^0_{\alpha\alpha}(\mathbf{r},\mathbf{r}',\omega) = \sum_{nn'} \frac{E_{n'}-E_n}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_\alpha\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_\alpha\Psi_{n'}(\mathbf{r}')\Big], \tag{2.23}$$

where we have used the relation between positive and negative energy states in Eq. (2.10). Similarly, the off-diagonal correlation functions $\Pi^0_{\alpha\neq\beta}$ involving coupling between fluctuations are given by

$$\Pi^0_{12}(\mathbf{r},\mathbf{r}',\omega) = -\sum_{nn'} \frac{\omega_+}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_2\Psi_{n'}(\mathbf{r}')\Big], \tag{2.24}$$

$$\Pi^0_{21}(\mathbf{r},\mathbf{r}',\omega) = -\sum_{nn'} \frac{\omega_+}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_2\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_1\Psi_{n'}(\mathbf{r}')\Big], \tag{2.25}$$

$$\Pi^0_{23}(\mathbf{r},\mathbf{r}',\omega) = -\sum_{nn'} \frac{\omega_+}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_2\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_3\Psi_{n'}(\mathbf{r}')\Big], \tag{2.26}$$

$$\Pi^0_{32}(\mathbf{r},\mathbf{r}',\omega) = -\sum_{nn'} \frac{\omega_+}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_3\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_2\Psi_{n'}(\mathbf{r}')\Big], \tag{2.27}$$

$$\Pi^0_{13}(\mathbf{r},\mathbf{r}',\omega) = \sum_{nn'} \frac{E_{n'}-E_n}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_3\Psi_{n'}(\mathbf{r}')\Big], \tag{2.28}$$

$$\Pi^0_{31}(\mathbf{r},\mathbf{r}',\omega) = \sum_{nn'} \frac{E_{n'}-E_n}{\omega_+^2-(E_{n'}-E_n)^2}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_3\Psi_n(\mathbf{r})\Big]\Big[\Psi^\dagger_n(\mathbf{r}')\sigma_1\Psi_{n'}(\mathbf{r}')\Big]. \tag{2.29}$$

When we substitute Eqs. (2.23)-(2.29) into Eq. (2.18), we obtain the following coupled equations for the collective modes

$$\begin{aligned}\delta\rho_1(\mathbf{r},\omega) = &-\frac{U}{2}\sum_{nn'} \frac{f_n - f_{n'}}{\omega_+^2-(E_{n'}-E_n)^2}\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})\Big]\Big[(E_{n'}-E_n)\langle n|\sigma_1\delta\rho_1(\mathbf{r}',\omega)|n'\rangle \\ &- \omega_+\langle n|\sigma_2\delta\rho_2(\mathbf{r}',\omega)|n'\rangle + (E_{n'}-E_n)\langle n|\sigma_3\delta\rho_3(\mathbf{r}',\omega)|n'\rangle\Big],\end{aligned} \tag{2.30}$$

$$\delta\rho_2(\mathbf{r},\omega) = -\frac{U}{2}\sum_{nn'}\frac{f_n - f_{n'}}{\omega_+^2 - (E_{n'}-E_n)^2}\Big[\Psi_{n'}^\dagger(\mathbf{r})\sigma_2\Psi_n(\mathbf{r})\Big]\Big[-\omega_+\langle n|\sigma_1\delta\rho_1(\mathbf{r}',\omega)|n'\rangle + (E_{n'}-E_n)\langle n|\sigma_2\delta\rho_2(\mathbf{r}',\omega)|n'\rangle - \omega_+\langle n|\sigma_3\delta\rho_3(\mathbf{r}',\omega)|n'\rangle\Big], \tag{2.31}$$

$$\delta\rho_3(\mathbf{r},\omega) = -\frac{U}{2}\sum_{nn'}\frac{f_n - f_{n'}}{\omega_+^2 - (E_{n'}-E_n)^2}\Big[\Psi_{n'}^\dagger(\mathbf{r})\sigma_3\Psi_n(\mathbf{r})\Big]\Big[(E_{n'}-E_n)\langle n|\sigma_1\delta\rho_1(\mathbf{r}',\omega)|n'\rangle - \omega_+\langle n|\sigma_2\delta\rho_2(\mathbf{r}',\omega)|n'\rangle + (E_{n'}-E_n)\langle n|\sigma_3\delta\rho_3(\mathbf{r}',\omega)|n'\rangle\Big]. \tag{2.32}$$

Here we have introduced the notation

$$\langle n|\sigma_\alpha\delta\rho_\alpha(\mathbf{r}')|n'\rangle \equiv \int d\mathbf{r}'\Psi_n^\dagger(\mathbf{r}')\sigma_\alpha\delta\rho_\alpha(\mathbf{r}')\Psi_{n'}(\mathbf{r}'). \tag{2.33}$$

In HFB-GRPA formalism, the collective modes are determined by the solutions of Eqs. (2.30)-(2.32). The single particle excitations E_n, particle density $n(\mathbf{r})$, and order parameter $\Delta(\mathbf{r})$ are obtained by solving Eqs. (2.7), (2.15) and (2.16) self-consistently.

C. Goldstone mode

Since we have assumed a short-range effective pairing interaction in Eq. (2.1), a cutoff ω_c is necessary in calculating correlation functions related to superfluid fluctuations, e.g., Π_{22}^0. A similar cutoff is needed in solving the gap equation (2.15). However, we must take some care in how we introduce this same cutoff $\omega_{\rm c}$ in calculating the correlation functions. For this purpose, a crucial role is played by involving Goldstone's theorem, which describes the appearance of an excitation in the ordered state associated with a broken continuous symmetry. We want our HFB-GRPA formalism to satisfy this theorem and thus it is reasonable to introduce ω_c to ensure this happens.

In the superfluid phase, the ground state is chosen from an infinitely large number of degenerate candidates for ground state, that are characterized by the phase of order parameter $\Delta(\mathbf{r})e^{i\phi}$ $(0 \le \phi < 2\pi)$. The Goldstone mode is the excitation from an assumed ground state (where we set $\phi = 0$) to the other degenerate states ($\phi \neq 0$)[39]. As a result, the Goldstone mode is physically described by a phase oscillation of the order parameter with a zero excitation energy. The phase fluctuations of order parameter described by $\delta\rho_2(\mathbf{r},\omega=0)$ in Eq. (2.31) are indeed decoupled from other fluctuations at $\omega = 0$, because one can show $\Pi_{12}^0 = \Pi_{21}^0 = \Pi_{23}^0 = \Pi_{32}^0 = 0$ (see Eqs. (2.24)-(2.27)). Taking $(\delta\rho_1,\delta\rho_2,\delta\rho_3) = (0,\delta\rho_2,0)$, we

are left with a single equation at $\omega = 0$,

$$\delta\rho_2(\mathbf{r},0) \;=\; -\frac{U}{2}\sum_{nn'}\frac{f_{n'}-f_n}{E_{n'}-E_n}\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_2\Psi_n(\mathbf{r})\Big]\langle n|\sigma_2\delta\rho_2(\mathbf{r}',0)|n'\rangle. \tag{2.34}$$

We introduce a cutoff ω_c in the energy-summation (n-summation) by attaching the step function $\Theta(\omega_c - |E_n|)$ to the Fermi distribution functions f_n and $f_{n'}$ in Eq. (2.34) as follows;

$$\begin{aligned} f_n &\;\rightarrow\; \tilde{f}_n \equiv \Theta(\omega_c - |E_n|)f_n, \\ f_{n'} &\;\rightarrow\; \tilde{f}_{n'} \equiv \Theta(\omega_c - |E_{n'}|)f_{n'}. \end{aligned} \tag{2.35}$$

To prove that this prescription is consistent with the BCS gap equation (2.15), we next show that $\delta\rho_2(\mathbf{r},0) = \Delta(\mathbf{r})$ is a solution of Eq. (2.34) *when the order parameter satisfies Eq. (2.15).* From the commutation relation $[\hat{h},\sigma_3] = -2i\Delta(\mathbf{r})\sigma_2$ (where $\hat{h}$ is defined in Eq. (2.8)), we obtain

$$\langle n|\sigma_2\Delta(\mathbf{r}')|n'\rangle = \frac{i}{2}\langle n|[\hat{h},\sigma_3]|n'\rangle = \frac{i}{2}(E_n - E_{n'})\langle n|\sigma_3|n'\rangle. \tag{2.36}$$

Substituting $\delta\rho_2(\mathbf{r},0) = \Delta(\mathbf{r})$ into the RHS of Eq. (2.34) ($\equiv S(\mathbf{r},0)$), we find

$$\begin{aligned} S(\mathbf{r},0) &= \frac{iU}{4}\sum_{nn'}[\tilde{f}_{n'} - \tilde{f}_n]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_2\Psi_n(\mathbf{r})\Big]\langle n|\sigma_3|n'\rangle \\ &= \frac{iU}{4}\sum_{n}\tilde{f}_n\Psi^\dagger_n(\mathbf{r})[\sigma_2,\sigma_3]\Psi_n(\mathbf{r}) \\ &= -\frac{U}{2}\sum_{n}\tilde{f}_n\Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r}) \\ &= -\frac{U}{2}\sum_{E_n>0}^{\omega_c}\Big[\Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})f_n + \Psi^\dagger_n(\mathbf{r})\sigma_2\sigma_1\sigma_2\Psi_n(\mathbf{r})(1-f_n)\Big] \\ &= \sum_{E_n>0}^{\omega_c}\Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})[1-2f_n]. \end{aligned} \tag{2.37}$$

In this calculation, we have used the completeness condition, $\sum_n \Psi_n(\mathbf{r})\Psi^\dagger_n(\mathbf{r}') = \delta(\mathbf{r}'-\mathbf{r})$. From the gap equation (2.15), Eq. (2.37) is found to be precisely equal to $\Delta(\mathbf{r})$.

As a result, we have shown that at $\omega = 0$,

$$\begin{pmatrix} \delta\rho_1(\mathbf{r},0) \\ \delta\rho_2(\mathbf{r},0) \\ \delta\rho_3(\mathbf{r},0) \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta(\mathbf{r}) \\ 0 \end{pmatrix}, \tag{2.38}$$

which describes the Goldstone mode. We conclude that Eq. (2.35) is consistent choice of the cutoff ω_c with the gap equation (2.15), in the sense that the HFB-GRPA leads to linear

response functions which satisfy Goldstone's theorem. In the following sections, we use Eq. (2.35) in evaluating the superfluid fluctuations described by $\delta\rho_1$ and $\delta\rho_2$ in Eqs. (2.30) and (2.31).

D. Kohn mode in Trapped Fermi superfluids

When the frequency of a collective mode is non-zero, density fluctuations couple with superfluid fluctuations, so that we have to solve the *coupled* equations (2.30)-(2.32). In this section, we prove that

$$\begin{pmatrix} \delta\rho_1(\mathbf{r},\Omega_x) \\ \delta\rho_2(\mathbf{r},\Omega_x) \\ \delta\rho_3(\mathbf{r},\Omega_x) \end{pmatrix} = \begin{pmatrix} \partial_x\Delta(\mathbf{r}) \\ -2im\Omega_x x\Delta(\mathbf{r}) \\ -\frac{U}{2}\partial_x n(\mathbf{r}) \end{pmatrix} \tag{2.39}$$

are the solutions of Eqs. (2.30)-(2.32), with frequency ω equal to the trap frequency Ω_x. We discuss the physical meaning of these solutions after giving the proof.

From the two commutation relations $[x,\hat{h}] = \frac{1}{m}\sigma_3\partial_x$ and $[\partial_x,\hat{h}] = \sigma_3 m\Omega_x^2 x + \sigma_1\partial_x\Delta(\mathbf{r}) - \frac{U}{2}\sigma_3\partial_x n(\mathbf{r})$, we obtain

$$(E_{n'} - E_n)\langle n|\sigma_3 x|n'\rangle = 2i\langle n|\sigma_2\Delta(\mathbf{r})x|n'\rangle + \frac{1}{m}\langle n|\partial_x|n'\rangle, \tag{2.40}$$

$$(E_{n'} - E_n)\langle n|\partial_x|n'\rangle = m\Omega_x^2\langle n|\sigma_3 x|n'\rangle + \langle n|\sigma_1(\partial_x\Delta(\mathbf{r}))|n'\rangle - \frac{U}{2}\langle n|\sigma_3(\partial_x n(\mathbf{r}))|n'\rangle. \tag{2.41}$$

These two equations give

$$\begin{aligned}\Big[(E_{n'} - E_n)^2 - \Omega_x^2\Big]\langle n|\partial_x|n'\rangle \;&=\; (E_{n'} - E_n)\langle n|\sigma_1(\partial_x\Delta(\mathbf{r}))|n\rangle \\ &+\; 2im\Omega_x^2\langle n|\sigma_2\Delta(\mathbf{r})x|n\rangle \\ &-\; (E_{n'} - E_n)\langle n|\sigma_3\frac{U}{2}(\partial_x n(\mathbf{r}))|n\rangle,\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}m\Big[(E_{n'} - E_n)^2 - \Omega_x^2\Big]\langle n|\sigma_3 x|n'\rangle \;&=\; \langle n|\sigma_1\Delta(\mathbf{r})x|n\rangle \\ &+\; 2im(E_{n'} - E_n)\langle n|\sigma_2\Delta(\mathbf{r})x|n\rangle \\ &-\; \langle n|\sigma_3\frac{U}{2}(\partial_x n(\mathbf{r}))|n\rangle.\end{aligned} \tag{2.43}$$

We next substitute Eq. (2.39) into the RHS of (2.30), which we denote as $S_1(\omega)$. At $\omega = \Omega_x$, S_1 can be reduced to, by using Eq. (2.42),

$$
\begin{aligned}
S_1(\Omega_x) &= \frac{U}{2}\sum_{nn'}[\tilde{f}_n - \tilde{f}_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})\Big]\langle n|\partial_{x'}|n'\rangle \\
&= -\frac{U}{2}\sum_n \tilde{f}_n\Big[(\partial_x\Psi^\dagger_n(\mathbf{r}))\sigma_1\Psi_n(\mathbf{r}) + \Psi^\dagger_n(\mathbf{r})\sigma_1\partial_x\Psi_n(\mathbf{r})\Big] \\
&= -\partial_x\frac{U}{2}\sum_n \tilde{f}_n\Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r}) \\
&= -\partial_x\frac{U}{2}\sum_{E_n>0}^{\omega_c}\Big[\Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})f_n + \Psi^\dagger_n(\mathbf{r})\sigma_2\sigma_1\sigma_2\Psi_n(\mathbf{r})(1-f_n)\Big] \\
&= \partial_x\frac{U}{2}\sum_{E_n>0}^{\omega_c}\Psi^\dagger_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})[1-2f_n] \\
&= \partial_x\Delta(\mathbf{r}) \quad (=\delta\rho_1(\mathbf{r},\Omega_x)).
\end{aligned}
\tag{2.44}
$$

Thus our trial solution in Eq. (2.39) satisfies Eq. (2.30) for $\omega = \Omega_x$.

In the same way, we also find that Eq. (2.39) is a solution of Eq. (2.32). Indeed, substituting Eq. (2.42) into the RHS of Eq. (2.32) with $\omega = \Omega_x$ ($\equiv S_3(\Omega_x)$), we obtain

$$
\begin{aligned}
S_3(\Omega_x) &= \frac{U}{2}\sum_{nn'}[f_n - f_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_3\Psi_n(\mathbf{r})\Big]\langle n|\partial_{x'}|n'\rangle \\
&= -\frac{U}{2}\partial_x\sum_n f_n\Psi^\dagger_n(\mathbf{r})\sigma_3\Psi_n(\mathbf{r}) \\
&= \frac{U}{2}\partial_x\sum_{E_n>0}\Psi^\dagger_n(\mathbf{r})\sigma_3\Psi_n(\mathbf{r})(1-2f_n) \\
&= -\frac{U}{2}\partial_x n(\mathbf{r}) \quad (\equiv\delta\rho_3(\mathbf{r},\Omega_x)).
\end{aligned}
\tag{2.45}
$$

The RHS of Eq. (2.31) ($\equiv S_2(\Omega_x)$) can be evaluated by using Eq. (2.43). We find, for $\omega = \Omega_x$,

$$
\begin{aligned}
S_2(\Omega_x) &= -\frac{U}{2}m\Omega_x\sum_{nn'}[\tilde{f}_n - \tilde{f}_{n'}]\Big[\Psi^\dagger_{n'}(\mathbf{r})\sigma_2\Psi_n(\mathbf{r})\Big]\langle n|\sigma_3 x|n'\rangle \\
&= iUm\Omega_x\sum_n \tilde{f}_n\Psi_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})x \\
&= -iUm\Omega_x\sum_{E_n>0}^{\omega_c}\Psi_n(\mathbf{r})\sigma_1\Psi_n(\mathbf{r})(1-2f_n) \\
&= -2im\Omega_x\Delta(\mathbf{r})x \quad (\equiv\delta\rho_2(\mathbf{r},\Omega_x)).
\end{aligned}
\tag{2.46}
$$

From the results in Eqs. (2.44)-(2.46), we have shown explicitly that fluctuations as given in Eq. (2.39) are indeed a solution of the coupled equations (2.30)-(2.32) with frequency

Ω_x. This is the Kohn mode. Our calculation shows that in the BCS pairing approximation, the Kohn mode at the trap frequency is described as a coupled oscillation of density, phase, and amplitude of order parameter. We note that this contrasts with the Goldstone mode at $\omega = 0$, which is associated with a pure phase oscillation.

In the study of BEC of trapped Bose gases, it has been shown that the Kohn mode is the rigid center of mass oscillation of the static condensate and non-condensate distributions[6]. We now discuss the physics of the Kohn mode in Fermi superfluids, as described by Eq. (2.39). In fermion superfluidity, the total local density profile at t is given by

$$n(\mathbf{r},t) = n(\mathbf{r}) + CRe[\delta\rho_3(\mathbf{r},\Omega_x)e^{i\Omega_x t}], \tag{2.47}$$

where C is a constant determining the amplitude of the oscillation. Substitute the third component of Eq. (2.39) into Eq. (2.47), we obtain

$$\begin{aligned} n(\mathbf{r},t) &= n(\mathbf{r}) - C\frac{U}{2}\partial_x n(\mathbf{r})\cos(\Omega_x t) \\ &\simeq n(\mathbf{r} - \mathbf{e}_x C\frac{U}{2}\cos(\Omega_x t)). \end{aligned} \tag{2.48}$$

Here $\mathbf{e}_x$ is the unit vector in the x-direction. The Kohn mode solution in Fermi superfluids is thus found to be the center of mass motion of the total density, just in the case in Bose gases[6]. Similarly, the oscillation of the order parameter is given by

$$\Delta(\mathbf{r},t) = \Delta(\mathbf{r}) + C\delta\Delta(\mathbf{r},t). \tag{2.49}$$

The second term is related the amplitude ($\delta\rho_1$) and phase oscillations ($\delta\rho_2$) of the order parameter, given by

$$\delta\Delta(\mathbf{r},t) = -\frac{U}{2}\Big[Re[\delta\rho_1(\mathbf{r},\Omega_x)e^{i\Omega_x t}] - iRe[\delta\rho_2(\mathbf{r},\Omega_x)e^{i\Omega_x t}]\Big]. \tag{2.50}$$

Using the first and second components in Eq. (2.39), we find (working to first order in the fluctuations)

$$\begin{aligned} \Delta(\mathbf{r},t) &= \Delta(\mathbf{r}) - C\frac{U}{2}\partial_x\Delta(\mathbf{r})\cos(\Omega_x t) + iCUmx\Omega_x\Delta(\mathbf{r})\sin(\Omega_x t) \\ &\simeq \Delta(\mathbf{r} - \mathbf{e}_x C\frac{U}{2}\cos(\Omega_x t))e^{iCUmx\Omega_x\sin(\Omega_x t)}. \end{aligned} \tag{2.51}$$

Since $|\Delta(\mathbf{r},t)|^2$ describes the density of the Cooper-pair condensate, we find that this condensate oscillates in the same way as the total density profile given by Eq. (2.48). Equation

(2.51) also shows that the Kohn mode is accompanied by a phase oscillation with the same frequency Ω_x. This is due to the Josephson effect, which couples the density oscillation with the phase oscillation of the Cooper-pair order parameter.

It is easy to show that the other two Kohn modes with the trap frequencies Ω_y and Ω_z are, respectively, given by

$$\begin{pmatrix} \delta\rho_1(\mathbf{r},\Omega_y) \\ \delta\rho_2(\mathbf{r},\Omega_y) \\ \delta\rho_3(\mathbf{r},\Omega_y) \end{pmatrix} = \begin{pmatrix} \partial_y\Delta(\mathbf{r}) \\ -2im\Omega_y y\Delta(\mathbf{r}) \\ -\frac{U}{2}\partial_y n(\mathbf{r}) \end{pmatrix}, \tag{2.52}$$

$$\begin{pmatrix} \delta\rho_1(\mathbf{r},\Omega_z) \\ \delta\rho_2(\mathbf{r},\Omega_z) \\ \delta\rho_3(\mathbf{r},\Omega_z) \end{pmatrix} = \begin{pmatrix} \partial_z\Delta(\mathbf{r}) \\ -2im\Omega_z z\Delta(\mathbf{r}) \\ -\frac{U}{2}\partial_z n(\mathbf{r}) \end{pmatrix}. \tag{2.53}$$

The above proof can be immediately extended to the strong-coupling theory developed by Nozières and Schmitt-Rink[24] at $T_{\rm c}$ in the context of superconductivity. In the NSR theory, fluctuations around the mean-field order parameter are taken into account within the Gaussian approximation[40]. One solves the gap equation together with the equation for the number of particles, which includes the effect of fluctuations around the mean field approximation. While the gap equation has the same form as Eq. (2.15) in the NSR theory, the chemical potential μ can be very different from the Fermi energy in the strong-coupling regime. When we consider collective modes in NSR-GRPA theory, we again obtain the linear response equations in Eqs. (2.30)- (2.32). The only difference is that the chemical potential is now determined by the equation for the number of particles, including the fluctuation effect in the Cooper channel. However, we note that the above proof based on Eqs. (2.30)-(2.32) is always valid irrespective of the value of the chemical potential. Thus, even if the chemical potential remarkably deviates from the Fermi energy due to the strong-coupling effect in the NSR-GRPA theory, we again obtain the Kohn modes with the frequencies Ω_α ($\alpha = x, y, z$), as well as the zero frequency Goldstone mode. Thus NSR-GRPA linear response formalism is found to be consistent with both Kohn's theorem and Goldstone's theorem, and can be used to study the BCS-BEC crossover, without violating these important theorems. This is a crucial requirement of any approximate many-body calculation of response functions.

III. KOHN MODE IN THE PRESENCE OF A FESHBACH RESONANCE

A. Coupled fermion-boson model

A Feshbach resonance can be used to produce a strong attractive (pairing) interaction in trapped Fermi gases[19, 20, 21, 22]. The Feshbach resonance is associated with a dimer bound state, which is a boson. This boson can enhance the pairing interaction between atoms. To include this effect in a simple way, the coupled fermion-boson model[10, 11, 12, 13, 14, 15, 16, 17, 18, 41, 42] is very convenient, which is given by

$$\begin{aligned} H &= \sum_\sigma \int d\mathbf{r}\Psi_\sigma^\dagger(\mathbf{r})[-\frac{\nabla^2}{2m} + V_{\rm trap} - \mu]\Psi_\sigma(\mathbf{r}) + \int d\mathbf{r}\Phi^\dagger(\mathbf{r})[-\frac{\nabla^2}{2M} + 2\nu + V^B_{\rm trap} - \mu_B]\Phi(\mathbf{r}) \\ &+ g_{\rm r}\int d\mathbf{r}\Big[\Phi^\dagger(\mathbf{r})\Psi_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}) + h.c.\Big] - U\int d\mathbf{r}\Psi_\uparrow^\dagger(\mathbf{r})\Psi_\downarrow^\dagger(\mathbf{r})\Psi_\downarrow(\mathbf{r})\Psi_\uparrow(\mathbf{r}). \end{aligned} \tag{3.1}$$

Here $\Phi(\mathbf{r})$ is a molecular Bose field operator describing molecules associated with the Feshbach resonance. $g_{\rm r}$ is the coupling constant of the Feshbach resonance, in which two Fermi atoms can form one Bose molecule. In turn, the molecule can dissociate into two Fermi atoms. 2ν is the threshold energy of the Feshbach resonance; the value of 2ν can be varied by an external magnetic field. In this model, U describes a non-resonant attractive interaction. Since one molecule consists of two Fermi atoms, we take $M = 2m$. We also impose the conservation of the total number of atoms as $N = N_{\rm F}+2N_{\rm B}$, where $N_{\rm F}$ and $N_{\rm B}$ represent the number of Fermi atoms and Bose molecules, respectively. This condition has been included in Eq. (3.1), with the chemical potentials μ and $\mu_B \equiv 2\mu$. $V^B_{\rm trap}$ is a harmonic trap potential for molecules, given by

$$V^B_{\rm trap} = \sum_{\alpha=x,y,z}\frac{1}{2}M{\Omega^B_\alpha}^2 r_\alpha^2. \tag{3.2}$$

If the molecules and atoms feel the same trap frequency, we have $\Omega^B_\alpha = \Omega_\alpha$. When the hyperfine states of atoms involved in the molecule are different from the atomic hyperfine states $\Psi_\sigma(\mathbf{r})$, the molecule may feel a different trap frequency.

Although the coupled fermion-boson model in Eq. (3.1) has the different form from Eqs. (1.1) and (2.1) due to the presence of molecular bosons, we can show that Kohn's theorem is exactly satisfied using Eq. (3.1) in the special case when $\Omega^B_\alpha = \Omega_\alpha$. The generators to excite the Kohn modes are now given by

$$\hat{P}_\alpha \equiv \sum_\sigma\int d\mathbf{r}\Psi_\sigma^\dagger(\mathbf{r})[m\Omega_\alpha\hat{r}_\alpha - i\hat{p}_\alpha]\Psi_\sigma(\mathbf{r})$$

$$+ \int d\mathbf{r}\Phi^\dagger(\mathbf{r})[M\Omega_\alpha^B \hat{r}_\alpha - i\hat{p}_\alpha]\Phi(\mathbf{r}). \quad (\alpha = x, y, z) \tag{3.3}$$

Using Eqs. (3.1) and (3.3), one can derive $[H, \hat{P}_\alpha] = \Omega_\alpha \hat{P}_\alpha$ $(\alpha = x, y, z)$. Thus $\hat{P}_\alpha|\Psi_0\rangle$ $(\alpha = 1, 2, 3)$ describe excited states with frequency Ω_α, when $|\Psi_0\rangle$ is the ground state wavefunction.

The HFB Hamiltonian for the coupled fermion-boson model in Eq. (3.1) has the form

$$H_{\rm HFB} = H_{\rm HFB}^F + H_{\rm HFB}^B. \tag{3.4}$$

Here the fermion Hamiltonian $H_{\rm HFB}^F$ has the same form as the BCS Hamiltonian in Eq. (2.4), except that the Cooper-pair order parameter $\Delta(\mathbf{r})$ is now replaced with the composite order parameter, [11, 12, 15, 17]

$$\tilde{\Delta}(\mathbf{r}) \equiv \Delta(\mathbf{r}) + g_{\rm r}\phi_m(\mathbf{r}), \tag{3.5}$$

consisting of the Cooper-pair $\Delta(\mathbf{r})$ plus the molecular condensate $\phi_m(\mathbf{r}) \equiv \langle\Phi(\mathbf{r})\rangle$. In the equilibrium state, these two order parameters are related to each other by the identity[44]

$$\frac{g_{\rm r}}{U}\Delta(\mathbf{r}) = \hat{h}_B\phi_m(\mathbf{r}), \tag{3.6}$$

where $\hat{h}_B \equiv -\frac{\nabla^2}{2M} + 2\nu + V_{\rm trap}^B(\mathbf{r}) - 2\mu$. The Bose Hamiltonian $H_{\rm HFB}^B$ is given by

$$H_{\rm HFB}^B = \int d\mathbf{r}\delta\Phi^\dagger(\mathbf{r})\hat{h}_B\delta\Phi(\mathbf{r}), \tag{3.7}$$

where $\delta\Phi(\mathbf{r}) \equiv \Phi(\mathbf{r}) - \langle\Phi(\mathbf{r})\rangle = \Phi(\mathbf{r}) - \phi_m$. The identity in Eq. (3.6) plays a crucial role in the following discussion.

The BdG equations have the same form as Eq. (2.7), except that now $\Delta(\mathbf{r})$ in Eq. (2.8) is replaced with the composite order parameter $\tilde{\Delta}(\mathbf{r})$. Using the solution of the BdG equations, the Cooper-pair order parameter $\Delta(\mathbf{r})$ and the Fermi atom density $n(\mathbf{r})$ are calculated from Eqs. (2.15) and (2.16), respectively. The equilibrium molecular condensate $\phi_m(\mathbf{r})$ is then obtained from Eq. (3.6).

The presence of molecular bosons leads to an effective interaction between Fermi atoms. In GRPA, this interaction introduces an additional term to the linear response equation (see Eq. (4.2) of Ref. [17])

$$\delta\hat{\rho}(\mathbf{r}, \omega) = \delta\hat{\rho}_U(\mathbf{r}, \omega) + \delta\hat{\rho}_{g_{\rm r}}(\mathbf{r}, \omega), \tag{3.8}$$

where

$$\delta\hat{\rho} \equiv \begin{pmatrix} \delta\rho_1 \\ \delta\rho_2 \\ \delta\rho_3 \end{pmatrix}. \tag{3.9}$$

In Eq. (3.8), $\delta\hat{\rho}_U(\mathbf{r},\omega)$ includes the contribution of non-resonant interaction, which has already appeared in Eqs. (2.30)-(2.32), given by

$$\delta\hat{\rho}_U(\mathbf{r},\omega) = -\frac{U}{2}\int d\mathbf{r}'\hat{\Pi}^0(\mathbf{r},\mathbf{r}',\omega)\delta\hat{\rho}(\mathbf{r}',\omega). \tag{3.10}$$

Here the matrix elements of $\hat{\Pi}^0(\mathbf{r},\mathbf{r}',\omega)$ are given by Eqs. (2.23)-(2.29), with $\Delta(\mathbf{r})$ being now replaced with $\tilde{\Delta}(\mathbf{r})$. The effect of Feshbach resonance appears in $\delta\hat{\rho}_{g_{\rm r}}(\mathbf{r},\omega)$ as

$$\delta\hat{\rho}_{g_{\rm r}}(\mathbf{r},\omega) = \frac{g_{\rm r}^2}{2}\int d\mathbf{r}'\int d\mathbf{r}''\hat{B}(\mathbf{r},\mathbf{r}',\omega)\hat{\Pi}^0(\mathbf{r}',\mathbf{r}'',\omega)\delta\hat{\rho}(\mathbf{r}'',\omega). \tag{3.11}$$

In contrast to the non-resonance part U in Eq. (3.10), the effective interaction described by $g_{\rm r}^2\hat{B}(\mathbf{r},\mathbf{r}',\omega)$ in Eq. (3.11) is *non-local and frequency-dependent*, reflecting that it is mediated by Bose excitations. The interaction kernel $\hat{B}(\mathbf{r},\mathbf{r}',\omega)$ is a 3×3-matrix, where only B_{11}, B_{12}, B_{21} and B_{22} are finite, indicating that it only works in the Cooper-channel. These matrix elements can be expressed in the form

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \hat{W}\hat{D}_0(\mathbf{r},\mathbf{r}',\omega)\hat{W}^\dagger, \tag{3.12}$$

where $\hat{W}$ is the unitary matrix,

$$\hat{W} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{3.13}$$

$\hat{D}_0(\mathbf{r},\mathbf{r}',\omega)$ is the matrix single-particle Green's function for a free Bose gas in a trap[43]. Using the eigenfunctions $\Phi_n(\mathbf{r})$ of $\hat{h}_B$ with energy

$$\xi_n = \sum_{\alpha=x,y,z}\Omega_\alpha^B\Big[n_\alpha + \frac{1}{2}\Big] + 2\nu - 2\mu, \tag{3.14}$$

we can write $\hat{D}_0(\mathbf{r},\mathbf{r}',\omega)$ as

$$\hat{D}_0(\mathbf{r},\mathbf{r}',\omega) = \sum_n \frac{\Phi_n(\mathbf{r})\Phi_n^*(\mathbf{r}')}{(\omega + i0^+)\sigma_3 - \xi_n}. \tag{3.15}$$

B. Kohn's theorem in the presence of a Feshbach resonance

In solving the coupled equations in Eq. (3.8) for a collective mode, it is useful to note that the contribution from the Feshbach resonance, $\delta\hat{\rho}_{g_{\rm r}}(\mathbf{r},\omega)$, does not affect the equation for $\delta\rho_3(\mathbf{r},\omega)$. Thus, we easily find from Eq. (2.45) that

$$\begin{pmatrix} \delta\rho_1(\mathbf{r},\Omega_x) \\ \delta\rho_2(\mathbf{r},\Omega_x) \\ \delta\rho_3(\mathbf{r},\Omega_x) \end{pmatrix} = \begin{pmatrix} \partial_x\tilde{\Delta}(\mathbf{r}) \\ -2im\Omega_x x\tilde{\Delta}(\mathbf{r}) \\ -\frac{U}{2}\partial_x n(\mathbf{r}) \end{pmatrix}, \tag{3.16}$$

is a solution of the equation for $\delta\rho_3$ when $\omega=\Omega_x$.

To see if Eq. (3.16) also satisfies the other two equations for $\delta\rho_1$ and $\delta\rho_2$ in Eq. (3.10), we substitute this *trial* solution into the first ($\equiv \delta\rho_{U1}$) and second ($\equiv \delta\rho_{U2}$) component of Eq. (3.10). These can be evaluated in the same way as in Eqs. (2.44) and (2.46), and we obtain at $\omega=\Omega_x$

$$\begin{pmatrix} \delta\rho_{U1} \\ \delta\rho_{U2} \end{pmatrix} = \begin{pmatrix} \partial_x\Delta(\mathbf{r}) \\ -2im\Omega_x x\Delta(\mathbf{r}) \end{pmatrix} = \frac{U}{g_{\rm r}} \begin{pmatrix} \partial_x(\hat{h}_B\phi_m(\mathbf{r})) \\ -2im\Omega_x x\hat{h}_B\phi_m(\mathbf{r}) \end{pmatrix} \equiv \frac{U}{g_{\rm r}}\delta\bar{\rho}_U(\mathbf{r},\Omega_x). \tag{3.17}$$

Here we have used the identity in Eq. (3.6).

Next we substitute the trial solution in Eq. (3.16) into the first ($\equiv \delta\rho_{g_{\rm r}1}$) and second ($\equiv \delta\rho_{g_{\rm r}2}$) component of Eq. (3.11), and set $\omega=\Omega_x$. Noting that the factor $\int d\mathbf{r}''\hat{\Pi}_0(\mathbf{r}',\mathbf{r}'',\Omega_x)\delta\hat{\rho}(\mathbf{r}'',\Omega_x)$ can be simplified by using Eq. (3.17), we find

$$\begin{pmatrix} \delta\rho_{g_{\rm r}1} \\ \delta\rho_{g_{\rm r}2} \end{pmatrix} = -g_{\rm r}\int d\mathbf{r}'\hat{W}\hat{D}_0(\mathbf{r},\mathbf{r}',\Omega_x)\hat{W}^\dagger\delta\bar{\rho}_U(\mathbf{r}',\Omega_x). \tag{3.18}$$

Substituting Eqs. (3.13), (3.15) and (3.17) into Eq. (3.18), we obtain

$$\delta\rho_{g_{\rm r}1}(\mathbf{r},\Omega_x) = g_{\rm r}\sum_n\frac{\Phi_n(\mathbf{r})}{\xi_n^2-\Omega_x^2}\int d\mathbf{r}'\Phi_n^*(\mathbf{r}')\Big[\xi_n\partial_{x'}(\hat{h}_B\phi_m(\mathbf{r}'))-M\Omega_x^2x'\hat{h}_B\phi_m(\mathbf{r}')\Big], \tag{3.19}$$

$$\delta\rho_{g_{\rm r}2}(\mathbf{r},\Omega_x) = -ig_{\rm r}\Omega_x\sum_n\frac{\Phi_n(\mathbf{r})}{\xi_n^2-\Omega_x^2}\int d\mathbf{r}'\Phi_n^*(\mathbf{r}')\Big[M\xi_n x'\hat{h}_B\phi_m(\mathbf{r}')-\partial_{x'}(\hat{h}_B\phi_m(\mathbf{r}'))\Big], \tag{3.20}$$

The operation of $\hat{h}_B$ on $\phi_m(\mathbf{r})$ can be conveniently calculated by expanding the molecular condensate wavefunction,

$$\phi_m(\mathbf{r}) = \sum_n \alpha_n \Phi_n(\mathbf{r}). \tag{3.21}$$

Here $\Phi_n(\mathbf{r})$ are the eigenfunctions of a non-interacting Bose gas of molecules in a trap [see Eq. (3.15)].

Putting all these together, we have

$$\delta\rho_{g_\mathrm{r}1}(\mathbf{r},\Omega_x) \;=\; g_\mathrm{r}\sum_{nn'}\alpha_{n'}\xi_{n'}\frac{\Phi_n(\mathbf{r})}{\xi_n^2-\Omega_x^2}\int d\mathbf{r}'\Phi_n^*(\mathbf{r}')\Big[\xi_n\partial_{x'}\Phi_{n'}(\mathbf{r}') - M\Omega_x^2 x'\Phi_{n'}(\mathbf{r}')\Big], \tag{3.22}$$

$$\delta\rho_{g_\mathrm{r}2}(\mathbf{r},\Omega_x) \;=\; -ig_\mathrm{r}\Omega_x\sum_{nn'}\alpha_{n'}\xi_{n'}\frac{\Phi_n(\mathbf{r})}{\xi_n^2-\Omega_x^2}\int d\mathbf{r}'\Phi_n^*(\mathbf{r}')\Big[M\xi_n x'\Phi_{n'}(\mathbf{r}') - \partial_{x'}\Phi_{n'}(\mathbf{r}')\Big]. \tag{3.23}$$

The integration over $\mathbf{r}'$ can be carried out by expressing x' and $\partial_{x'}$ in terms of raising and lowering operator of a harmonic oscillator for a molecule,

$$\hat{a}^\dagger \equiv \frac{1}{\sqrt{2M\Omega_x^B}}[-\partial_x + M\Omega_x^B x], \tag{3.24}$$

$$\hat{a} \equiv \frac{1}{\sqrt{2M\Omega_x^B}}[\partial_x + M\Omega_x^B x]. \tag{3.25}$$

Since $\hat{a}\Phi_n(\mathbf{r}) = \sqrt{n_x}\Phi_{n-1}(\mathbf{r})$ and $\hat{a}^\dagger\Phi_n(\mathbf{r}) = \sqrt{n_x+1}\Phi_{n+1}(\mathbf{r})$ (where $n\pm1$ is an abbreviation for $(n_x\pm1, n_y, n_z)$), Eqs. (3.22) and (3.23) reduce to, respectively,

$$\begin{aligned}\delta\rho_{g_\mathrm{r}1}(\mathbf{r},\Omega_x) \;&=\; \sqrt{\frac{M\Omega_x^B}{2}}g_\mathrm{r}\sum_{nn'}\alpha_{n'}\xi_{n'}\frac{\Phi_n(\mathbf{r})}{\xi_n^2-\Omega_x^2}\int d\mathbf{r}'\Phi_n^*(\mathbf{r}')\Big[(\xi_n-\Omega_x)\hat{a} - (\xi_n+\Omega_x)\hat{a}^\dagger\Big]\Phi_{n'}(\mathbf{r}')\\ &=\; \sqrt{\frac{M\Omega_x^B}{2}}g_\mathrm{r}\Big[\sum_n\sqrt{n_x+1}\alpha_{n+1}\frac{\xi_{n+1}}{\xi_n+\Omega_x}\Phi_n(\mathbf{r}) - \sum_{n\ (n_x\geq1)}\sqrt{n_x}\alpha_{n-1}\frac{\xi_{n-1}}{\xi_n-\Omega_x}\Phi_n(\mathbf{r})\Big],\end{aligned} \tag{3.26}$$

$$\begin{aligned}\delta\rho_{g_\mathrm{r}2}(\mathbf{r},\Omega_x) \;&=\; -i\sqrt{\frac{M}{2\Omega_x^B}}\Omega_x g_\mathrm{r}\sum_{nn'}\alpha_{n'}\xi_{n'}\frac{\Phi_n(\mathbf{r})}{\xi_n^2-\Omega_x^2}\int d\mathbf{r}'\Phi_n^*(\mathbf{r}')\Big[(\xi_n-\Omega_x)\hat{a} + (\xi_n+\Omega_x)\hat{a}^\dagger\Big]\Phi_{n'}(\mathbf{r}')\\ &=\; -i\sqrt{\frac{M}{2\Omega_x^B}}\Omega_x g_\mathrm{r}\Big[\sum_n\sqrt{n_x+1}\alpha_{n+1}\frac{\xi_{n+1}}{\xi_n+\Omega_x}\Phi_n(\mathbf{r}) + \sum_{n\ (n_x\geq1)}\sqrt{n_x}\alpha_{n-1}\frac{\xi_{n-1}}{\xi_n-\Omega_x}\Phi_n(\mathbf{r})\Big].\end{aligned} \tag{3.27}$$

When the trap frequency for molecules Ω_x^B is equal to that of Fermi atoms Ω_x, the factors $\xi_{n\pm1}/(\xi_n \pm \Omega_x)$ disappears because of $\xi_{n\pm1} \equiv \xi_n \pm \Omega_x^B = \xi_n \pm \Omega_x$. In this particular case, we find

$$
\begin{aligned}
\delta\rho_{g_{\rm r}1}(\mathbf{r},\Omega_x) &= \sqrt{\frac{M\Omega_x}{2}} g_{\rm r} \sum_n \sqrt{n_x+1}\alpha_{n+1}\Phi_n(\mathbf{r}) - \sqrt{\frac{M\Omega_x}{2}} g_{\rm r} \sum_{n\ (n_x\geq 1)} \sqrt{n_x}\alpha_{n-1}\Phi_n(\mathbf{r}), \\
&= g_{\rm r}\sqrt{\frac{M\Omega_x}{2}}[\hat{a}-\hat{a}^\dagger]\sum_n \alpha_n \Phi_n(\mathbf{r}) \\
&= g_{\rm r}\partial_x\phi_m(\mathbf{r}),
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
\delta\rho_{g_{\rm r}2}(\mathbf{r},\Omega_x) &= -i\sqrt{\frac{M\Omega_x}{2}} g_{\rm r} \sum_n \sqrt{n_x+1}\Phi_n(\mathbf{r}) - i\sqrt{\frac{M\Omega_x}{2}} g_{\rm r} \sum_{n\ (n_x\geq 1)} \sqrt{n_x}\alpha_{n-1}\Phi_n(\mathbf{r}) \\
&= -ig_{\rm r}\sqrt{\frac{M\Omega_x}{2}}[\hat{a}+\hat{a}^\dagger]\sum_n \alpha_n \Phi_n(\mathbf{r}) \\
&= -2img_{\rm r}\Omega_x x\phi_m(\mathbf{r}).
\end{aligned}
\tag{3.29}
$$

From Eqs. (3.17), (3.28) and (3.29), we find that the upper two components of the RHS of Eq. (3.8) are described by the composite order parameter $\tilde{\Delta}(\mathbf{r}) = \Delta(\mathbf{r}) + g_{\rm r}\phi_m(\mathbf{r})$ as

$$
\begin{pmatrix} \delta\rho_{U1} + \delta\rho_{g_{\rm r}1} \\ \delta\rho_{U2} + \delta\rho_{g_{\rm r}2} \end{pmatrix} = \begin{pmatrix} \partial_x\tilde{\Delta}(\mathbf{r}) \\ -2im\Omega_x x\tilde{\Delta}(\mathbf{r}) \end{pmatrix}. \tag{3.30}
$$

Thus we have proven that Eq. (3.16) is a solution of the collective mode equation (3.8) with frequency $\omega = \Omega_x$ (in the special case when $\Omega_x^B = \Omega_x$). We can also show that

$$
\begin{pmatrix} \delta\rho_1(\mathbf{r},\Omega_y) \\ \delta\rho_2(\mathbf{r},\Omega_y) \\ \delta\rho_3(\mathbf{r},\Omega_y) \end{pmatrix} = \begin{pmatrix} \partial_y\tilde{\Delta}(\mathbf{r}) \\ -2im\Omega_y y\tilde{\Delta}(\mathbf{r}) \\ -\frac{U}{2}\partial_y n(\mathbf{r}) \end{pmatrix}, \tag{3.31}
$$

$$
\begin{pmatrix} \delta\rho_1(\mathbf{r},\Omega_z) \\ \delta\rho_2(\mathbf{r},\Omega_z) \\ \delta\rho_3(\mathbf{r},\Omega_z) \end{pmatrix} = \begin{pmatrix} \partial_z\tilde{\Delta}(\mathbf{r}) \\ -2im\Omega_z z\tilde{\Delta}(\mathbf{r}) \\ -\frac{U}{2}\partial_z n(\mathbf{r}) \end{pmatrix}, \tag{3.32}
$$

are also the solutions of Eq. (3.8) with frequency $\omega = \Omega_y$ and Ω_z, respectively. These explicit solutions describe the Kohn mode in the coupled Fermion-Boson model.

We now briefly discuss the physical meaning of the Kohn mode in the presence of a Feshbach resonance, given by Eq. (3.16). The density oscillation (see $\delta\rho_3(\mathbf{r},\Omega_x)$ in Eq.(3.16))

is found to have the same form as Eq. (2.48) obtained in the absence of the Feshbach resonance. On the other hand, the oscillation of the Cooper-pair order parameter,

$$\Delta(\mathbf{r},t) = \Delta(\mathbf{r}) - C\frac{U}{2}\partial_x\tilde{\Delta}(\mathbf{r})\cos(\Omega_x t) + iCUmx\Omega_x\tilde{\Delta}(\mathbf{r})\sin(\Omega_x t) \tag{3.33}$$

cannot be written in the form given in Eq. (2.51), because the *composite order parameter* appears in the RHS of Eq. (3.33). This reflects the fact that the Cooper-pair oscillations are strongly coupled with molecular Bose excitations through the Feshbach resonance. Indeed, the linear response of the Bose condensate $\phi_m(\mathbf{r})$ induced by the oscillation of Fermi atoms described by Eq. (3.16) is given by

$$\langle\delta\Phi(\mathbf{r},\Omega_x)\rangle = \frac{g_{\rm r}}{2}\int d\mathbf{r}' D^0_{11}(\mathbf{r},\mathbf{r}',\Omega_x)[\delta\rho_1(\mathbf{r}',\Omega_x) - i\delta\rho_2(\mathbf{r}',\Omega_x)]. \tag{3.34}$$

Here $D^0_{11}(\mathbf{r},\mathbf{r}',\Omega_x)$ is the diagonal component of molecular Bose Green's function defined in Eq. (3.14). Substituting Eq. (3.16) (which we multiply with the factor C) and Eq. (3.14) into Eq. (3.34), and using the same method used in Eqs. (3.26)-(3.29), we obtain (when $\Omega^B_x = \Omega_x$)

$$\langle\delta\Phi(\mathbf{r},\Omega_x)\rangle = -C\frac{g_{\rm r}}{2}[\partial_x - M\Omega_x x]\hat{h}_B^{-1}\tilde{\Delta}(\mathbf{r}), \tag{3.35}$$

where the molecular Hamiltonian $\hat{h}_B$ is defined just before Eq. (3.7). In the same way, we also obtain

$$\langle\delta\Phi^\dagger(\mathbf{r},\Omega_x)\rangle = -C\frac{g_{\rm r}}{2}[\partial_x + M\Omega_x x]\hat{h}_B^{-1}\tilde{\Delta}(\mathbf{r}). \tag{3.36}$$

As a result, the oscillation of the Bose condensate associated with the Kohn mode is given by

$$\phi_m(\mathbf{r},t) = \phi_m(\mathbf{r}) - C\frac{g_{\rm r}}{2}\partial_x(\hat{h}_B^{-1}\tilde{\Delta}(\mathbf{r}))\cos(\Omega_x t) + iCm\Omega_x x g_{\rm r}\hat{h}_B^{-1}\tilde{\Delta}(\mathbf{r})\sin(\Omega_x t). \tag{3.37}$$

Putting Eqs (3.33) and (3.37) together, the oscillation of the *composite* order parameter $\tilde{\Delta}(\mathbf{r},t) = \Delta(\mathbf{r},t) + g_{\rm r}\phi_m(\mathbf{r},t)$ is given by

$$\tilde{\Delta}(\mathbf{r},t) = \tilde{\Delta}(\mathbf{r}) - \frac{C}{2}\partial_x\Big([U + \frac{g_r^2}{\hat{h}_B}]\tilde{\Delta}(\mathbf{r})\Big)\cos(\Omega_x t) + iCm\Omega_x x[U + \frac{g_r^2}{\hat{h}_B}]\tilde{\Delta}(\mathbf{r}). \tag{3.38}$$

We note that $U + \frac{g_r^2}{\hat{h}_B}$ describes the interaction between atoms, where $\frac{g_r^2}{\hat{h}_B}$ involves the dynamical effect associated with the Feshbach resonance. Indeed, in a uniform Fermi gas, when one neglects the kinetic energy of Bose molecules, this factor reduces to $U + g_{\rm r}/(2\nu - 2\mu)$, which has been previously obtained as the pairing interaction associated with the Feshbach

resonance [11, 12, 15, 16, 17]. Using Eqs. (3.6) and (3.21), we can expand the equilibrium composite order parameter in terms of the eigenfunctions of the Bose molecules $\Phi_n(\mathbf{r})$ in a trap,

$$\begin{aligned}\tilde{\Delta}(\mathbf{r}) &= g_{\rm r}\sum_n \alpha_n\Big(1+\frac{U}{g_{\rm r}^2}\xi_n\Big)\Phi_n(\mathbf{r}) \\ &\equiv \sum_n \beta_n\Phi_n(\mathbf{r}).\end{aligned} \tag{3.39}$$

Substituting Eq. (3.39) into Eq. (3.38), we find

$$\begin{aligned}\tilde{\Delta}(\mathbf{r},t) &= \sum_n \beta_n\Phi_n(\mathbf{r}) - \frac{C}{2}\sum_n \beta_n\Big[U+\frac{g_{\rm r}^2}{\xi_n}\Big]\partial_x\Phi_n(\mathbf{r})\cos(\Omega_x t) \\ &+ iCm\Omega_x x\sum_n \beta_n\Big[U+\frac{g_{\rm r}^2}{\xi_n}\Big]\Phi_n(\mathbf{r})\sin(\Omega_x t) \\ &\simeq \sum_n \beta_n\Phi_n\Big(\mathbf{r}-\mathbf{e}_x\frac{C}{2}[U+\frac{g_{\rm r}^2}{\xi_n}]\cos(\Omega_x t)\Big)e^{iCm\Omega_x x(U+g_{\rm r}^2/\xi_n)\sin(\Omega_x t)}.\end{aligned} \tag{3.40}$$

Equation (3.40) shows that each eigenfunction $\Phi_n(\mathbf{r})$ rigidly oscillates around the center of mass with the frequency Ω_x. However, since the amplitude of this oscillation in each component (which is given by $\frac{C}{2}[U+\frac{g_{\rm r}^2}{\xi_n}]$) is different, the oscillation of the composite order parameter, which is given by $|\tilde{\Delta}(\mathbf{r},t)|^2$, is not described as $|\tilde{\Delta}(\mathbf{r}+\mathbf{e}_x C'\cos(\Omega_x t))|^2$. This is quite different from the Kohn mode in a Fermi superfluid in the absence of a Feshbach resonance [see Eq.(2.51)]. This difference arises because the effective interaction associated with the Feshbach resonance involves the *dynamical* effect of molecular Bosons, so that the interaction between atoms depends on energy as $U+\frac{g_{\rm r}^2}{\xi_n}$. In the BEC limit, where the composite order parameter is described by the ground state of the harmonic potential ($\Phi_0(\mathbf{r})$), only the component with $n=0$ remains in Eq. (3.40). In this limiting case, Eq. (3.40) does reduce to $|\tilde{\Delta}(\mathbf{r},t)|^2 = |\tilde{\Delta}(\mathbf{r}+\mathbf{e}_x C'\cos(\Omega_x t))|^2$. Thus in this limit, we arrive at the usual Kohn mode solutions for a Bose condensed gas, involving an oscillation of the equilibrium order parameter.

We might note that the Kohn mode solutions which appear in the linear response functions describing the Fermi fields also appear in the Bose excitation spectrum. This is to be expected, since in the presence of a Feshbach resonance, one knows[15, 16, 17] that the collective modes of the fermions associated with Cooper-pairs are strongly coupled to the molecular excitations. The 2×2-matrix renormalized Bose Green's function $\hat{D}$ in the HFB-GRPA is given by[17, 42]

$$\hat{D}(\omega) = \Big[1-\hat{\Sigma}(\omega)\hat{D}^0(\omega)\Big]^{-1}\hat{D}^0(\omega). \tag{3.41}$$

Here we have used matrix notation for the dependence on $\mathbf{r}$ [see Eq. (3.11)]. The self-energy correction $\hat{\Sigma}(\omega)$ includes the effect of fluctuations in the Fermi atoms, given by

$$\hat{\Sigma}(\omega) = \frac{g_{\rm r}^2}{2}\hat{W}^\dagger\hat{\eta}\Big[\hat{\Pi}^0(\omega)\big[1+\frac{U}{2}\hat{\Pi}^0(\omega)\big]^{-1}\Big]\hat{W}, \tag{3.42}$$

where the projection operator $\hat{\eta}[\hat{A}]$ extracts the (11), (12), (21), and (22) components from a 3×3-matrix $\hat{A}$. The molecular Bose excitation spectrum is determined from the poles of Eq. (3.41), given by the condition that the determinant vanishes,

$$\begin{aligned} 0 &= det\Big[1-\frac{g_{\rm r}^2}{2}\hat{D}^0(\omega)\hat{W}^\dagger\hat{\eta}\Big[\hat{\Pi}^0(\omega)\big[1+\frac{U}{2}\hat{\Pi}^0(\omega)\big]^{-1}\Big]\hat{W}\Big] \\ &= \frac{det\Big[1+\frac{1}{2}\big[U-g_{\rm r}^2\hat{B}(\omega)\big]\hat{\Pi}^0(\omega)\Big]}{det\Big[1+\frac{U}{2}\hat{\Pi}^0(\omega)\Big]}. \end{aligned} \tag{3.43}$$

In the last expression, the determinant is taken over 3×3-matrices in the $(\hat{\rho}_1,\hat{\rho}_2,\hat{\rho}_3)$-space. Equation (3.43) always has solutions corresponding to the poles of the Fermi linear response functions. Thus, the renormalized Bose Green's functions in Eq. (3.41) also exhibit the Kohn mode solutions at $\omega=\Omega_\alpha$ (when $\Omega_\alpha^B=\Omega_\alpha$).

As discussed in Sec. II, the extension to include the strong-coupling effect based on the NSR theory does not destroy our proof. Thus, we can safely study collective modes in the BCS-BEC crossover region by using the NSR-GRPA formalism even in the presence of the Feshbach resonance, without any breakdown of Kohn's theorem, as long as the atomic and molecular trap frequencies are identical ($\Omega_\alpha^B=\Omega_\alpha$).

When the molecules and atoms have different trap frequencies ($\Omega_\alpha^B\neq\Omega_\alpha$), the key solutions in Eqs. (3.28) and (3.29) are no longer satisfied. In this regard, we recall that, in the presence of a Feshbach resonance, the dominant particles continuously change from unpaired Fermi atoms to molecular bosons as one goes through the strong-coupling BEC regime (i.e., decrease the threshold energy 2ν[15, 16, 17, 18, 44]). As a result, the *average* trap frequency which the dominant particles feel also changes from Ω_α to Ω_α^B, which in turn must affect the frequency of the "Kohn mode" in the BCS-BEC crossover. The breakdown of Kohn's theorem when $\Omega_\alpha^B\neq\Omega_\alpha$ is not due to the approximation used in HFB-GRPA (or NSR-GRPA), but rather is due to the changing nature of the excitation spectrum peculiar to a trapped Fermi gas with a Feshbach resonance.

Fig.1 shows the calculated frequency of the dipole mode at $T=0$ in the BCS-BEC crossover given by an approximate theory. We find that Kohn's theorem holds well (within

our numerical accuracy) when $\Omega_B = \Omega_F$. On the other hand, the mode frequency *depends on* the threshold energy 2ν in the BCS-BEC crossover when $\Omega_B \neq \Omega_F$. It continuously changes from the trap frequency of Fermi atoms Ω_F to that of Bose molecules Ω_B as one passes through the BCS-BEC crossover regime, as one expects.

C. Goldstone's theorem in the presence of a Feshbach resonance

In this final subsection, we briefly discuss the zero frequency Goldstone mode in the coupled Fermion-Boson model. At $\omega = 0$, the interaction kernel $\hat{B}$ in Eq. (3.12) is proportional to the unit matrix, so that, as in the BCS model discussed in Sec.II C, we need only consider the phase fluctuation component $\delta\rho_2$ in Eq. (3.8). This has the form

$$\delta\rho_2(\mathbf{r},0) = -\frac{U}{2}\int d\mathbf{r}'\Pi^0_{22}(\mathbf{r},\mathbf{r}',0)\delta\rho_2(\mathbf{r}',0) + \frac{g_{\rm r}^2}{2}\int d\mathbf{r}'\int d\mathbf{r}'' D^0_{22}(\mathbf{r},\mathbf{r}',0)\Pi^0_{22}(\mathbf{r}',\mathbf{r}'',0). \tag{3.44}$$

When we take $\delta\rho_2(\mathbf{r},0) = \tilde{\Delta}(\mathbf{r})$, we find following the discussion in Sec. II.C that the first term in the RHS of this equation reduces to $\Delta(\mathbf{r})$. Using the same method used to derive Eq. (3.18), we can integrate over $\mathbf{r}''$ in the second term of Eq. (3.44) ($\equiv \delta\rho_{g_{\rm r}2}$) to obtain

$$\delta\rho_{g_{\rm r}2} \;=\; -g_{\rm r}\int d\mathbf{r}' D^0_{22}(\mathbf{r},\mathbf{r}',0)\hat{h}_B\phi_m(\mathbf{r}'). \tag{3.45}$$

Using Eqs. (3.15) (with $\omega = 0$) and (3.21), we find

$$\begin{aligned}\delta\rho_{g_{\rm r}2} \;&=\; g_{\rm r}\sum_{nn'}\alpha_{n'}\frac{\xi_{n'}}{\xi_n}\Phi_n(\mathbf{r})\int d\mathbf{r}'\Phi^*_n(\mathbf{r}')\Phi_{n'}(\mathbf{r}')\\ &=\; g_{\rm r}\sum_n \alpha_n\Phi_n(\mathbf{r})\\ &=\; g_{\rm r}\phi_m(\mathbf{r}).\end{aligned} \tag{3.46}$$

Thus the RHS of Eq. (3.44) has been shown explicitly to equal $\Delta(\mathbf{r}) + g_{\rm r}\phi_m(\mathbf{r}) \equiv \tilde{\Delta}(\mathbf{r})$. This proves that $\delta\rho_2(\mathbf{r},0) = \tilde{\Delta}(\mathbf{r})$ describes the zero frequency Goldstone mode. In contrast to the BCS model in the absence of a Feshbach resonance, however, the Goldstone mode is now a collective phase oscillation of the *composite* order parameter, including contributions associated with both Cooper-pairs and the molecular condensate. We conclude that the HFB-GRPA (and NSR-GRPA) formalism is consistent with Goldstone's theorem even in the presence of the Feshbach resonance. We note that Goldstone mode arises even if the

trap frequencies felt by Fermi atoms and Bose molecules are different. We also note that our result also guarantees that the renormalized Bose Green's function in Eq. (3.41) has a gapless (zero frequency) excitation, a required condition for any approximate theory used to study Bose condensation.

IV. SUMMARY

In this paper, we have proved that the HFB-GRPA is a consistent formalism with Kohn's theorem at all temperatures. This proof is also valid for the strong-coupling superfluid theory developed by Nozières and Schmitt-Rink, as used in Ref.[17]. Using the NSR-GRPA formalism, we can safely study the linear response dynamics of the superfluid phase in the BCS-BEC crossover without breakdown of this general theorem on the dipole oscillation. The relevance of the Kohn mode in BCS superfluids was never discussed much in the context of superconductivity. In the case of superfluid Fermi gases trapped in a parabolic potential, it is important that any approximate theory used to calculate collective modes be consistent with Kohn's theorem.

We also considered, for the first time, the effect of a Feshbach resonance and the associated formation of molecules on the Kohn mode. When the molecules feel the same trap frequency as that for Fermi atoms, we explicitly proved that HFB-GRPA and NSR-GRPA lead to Kohn's theorem. However, when the molecular trap frequency is different from the atomic trap frequency (which can arise when dealing with different hyperfine states), the dipole mode frequency depends on the strength of the effective interaction (through 2ν) associated with the Feshbach resonance in the BCS-BEC crossover region. This was to be expected, of course, since the dominant excitations continuously change from Fermi atoms to Bose molecules as we go through the crossover regime (i.e., decrease the base molecular threshold 2ν). This result is a clear experimental signature of different trap frequencies.

We also have given, for the first time, a detailed discussion of the various quantities which are oscillating in the Kohn mode in a trapped superfluid Fermi gas, as given by our explicit results in Eqs. (2.39) and (3.16). In particular, we showed that with a Feshbach resonance, the Kohn mode involves a much more complex oscillation [see discussion after Eq. (3.33)] than without a Feshbach resonance [see discussion after Eq. (2.47)].

The BCS-BEC crossover and the Feshbach resonance are key phenomena in current stud-

ies on superfluidity in ultra-cold Fermi gases, such as ^{40}K and ^{6}Li. In considering these phenomena, one should be careful to introduce approximations which do not break Kohn's theorem, which is an exact result of a many-body system in a harmonic trap. Our proof on the consistency of HFB-GRPA and NSR-GRPA with both Kohn's theorem and Goldstone's theorem shows that they can be used to study the collective modes of *strongly-correlated* superfluid Fermi gases. Results of such calculations will be reported elsewhere[45].

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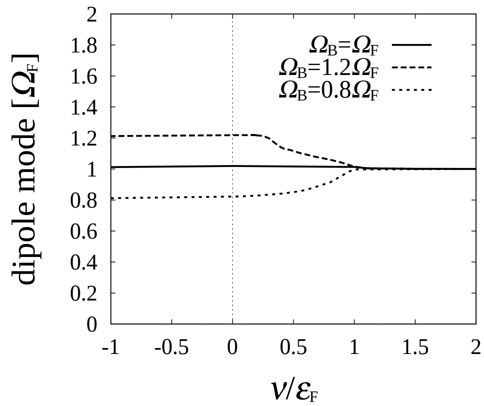

FIG. 1: Calculated frequency of the dipole mode at $T = 0$ in the superfluid phase of a trapped Fermi gas with a Feshbach resonance. We consider an isotropic harmonic trap with $\Omega_\alpha =\equiv \Omega_F$ and $\Omega_\alpha^B \equiv \Omega_B$. The Fermi energy $\varepsilon_{\rm F} = 31.5\Omega_F$ is for a free Fermi gas, with $N = 10,912$ atoms. We take $U = 0.001\Omega_F$, $g_{\rm r} = 0.06\Omega_F$ (which gives $UN/R_{\rm F}^3 = 0.35\varepsilon_{\rm F}$ and $g_{\rm r}\sqrt{N}/R_{\rm F}^{3/2} = 0.2\varepsilon_{\rm F}$, where $R_{\rm F} \equiv \sqrt{2\varepsilon_{\rm F}/m\Omega_F^2}$ is the Thomas-Fermi radius), and $\omega_c = 161.5\Omega_F$.